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***Abstract*— For a particular family of systems, we construct observers in the case where the measured variables are affected by the presence of a point-wise time-varying delay. The key feature of the proposed observers is that the size of their gains is proportional to the inverse of the largest value taken by the delay. The main result is first presented in the case of linear chain of integrators and next is extended to nonlinear systems with specific nonlinearities. A numerical example is provided to show the validity and effectiveness of the proposed observers.**

***Index Terms*— Observer design; time-delay; feedforward systems.**

I. INTRODUCTION

Time-delay systems have attracted the attention of many researchers in the field of control design. They become increasingly a subject of research activities over the years, as illustrated for instance by the survey [1], the works on observers [2], [3], [4] and [5], contributions on static output feedback stabilization [6] and the research monographs [7] and [8]. This is mainly due to the fact that time delays are frequently encountered in engineering applications (notably in aerospace systems, marine robotics, network control, population dynamics, [9]) and may significantly affect the performances of control laws and observers. Moreover, the great interest to the analysis and synthesis of control laws and observers for time-delay systems is also motivated by difficult challenges they present, from a mathematical point of view [2], [10], [11]. This is the case in particular of the problem of designing an observer in the presence of time-varying delay in the output measurements. Some important results on this subject are available in the literature. Since the pioneer work in [12], many extensions and improvements have been proposed. In [13], the authors proposed a delay dependent state observer that allows small delays. This technique has been revisited in [14] by using a chain observer to deal with arbitrarily long delay in the measurements. An improvement has been proposed in [15] by using cascade observer, where the head of the cascade is a high-gain observer while the remaining systems are state predictors. However, the use of high-gain observer may reduce the size of the allowed delay in the output measurement and it turns out that the delay are in some cases large, notably when they are caused by transport phenomena. This motivates the present work, where we propose an entirely new technique for several different classes of systems, which include the fundamental case of the chains of integrators and the family of the feedforward systems which includes systems of great relevance from an applied point of view such as Euler-Lagrange systems [16] and nonholonomic systems [17]. The procedure we propose is based on the use of different tools to deal with an arbitrarily long delay and uses small gain results. Let us describe now more precisely what is our contribution.

We tackle the problem of observer design for linear and nonlinear systems despite the presence of a time-varying pointwise delay in the output measurements. The fact our assumptions allow the delay to be time-varying implies that classical prediction based approaches (see in particular [9] Chapter 6) do not apply. We construct an observer for families of systems in the case where there is an arbitrarily large time-varying delay in the measured variables. First, we investigate the case of linear chains of integrators, whose importance is significant because they are present in the Jordan form of many systems and can be stabilized by static output feedback in the presence of arbitrarily large constant delays [6]. Next we extend the result to a class of nonlinear feedforward systems, which are used to model systems such as the cart-pendulum system and non-holonomic vehicles, which have been extensively studied, notably in the presence of delay, [18], [19], [20]. We borrow from [21] technical tools to propose a new observer design, which relies on *small gains*. It is worth noticing that our approach is different from the one of [14]. Indeed, the main feature of the proposed design is that the size of the observer gains is proportional to the inverse of the largest value taken by the delay. The idea is based on the use of the time-rescaling technique [21], which allows exponential convergence for arbitrarily long delay in the output measurements.

The paper is organized as follows. The main idea on the observer design problem with arbitrarily long delay in the output measurements for linear chain of integrators is introduced in Section II. Section III gives an extension of the small gain approach to a class of nonlinear systems. A numerical example is provided in Section IV to illustrate the proposed method. Finally, we end the paper by a conclusion in Section V.

II. OBSERVER DESIGN WITH DELAY IN THE OUTPUT

This section is devoted to the design of an exponentially convergent observer for a linear system, despite the presence of an arbitrarily long delay in the output measurements.

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A. System description and observer design

In this part, we recall a robustness result with respect to the presence of a small delay for linear observers. We consider the system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + G(t)u(t) \\ y(t) & = & Cx(t-\tau(t)), \end{array} \tag{1}$$

where the pair (A, C) is detectable and where $G(.)$ is a piecewise continuous function. We assume that τ is piecewise continuous and that there is $\overline{\tau} > 0$ such that $\tau(t) \in [0, \overline{\tau}]$ for all $t \geq 0$.

Since (A, C) is detectable, there is a matrix L such that the matrix $A+LC$ is Hurwitz. Let us introduce the following candidate observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + G(t)u(t) + L[y(t) - C\hat{x}(t-\tau(t))]. \tag{2}$$

To analyze whether the system (2) is an observer for (1), the convergence properties of this candidate observer, we introduce the error variable $z = x(t) - \hat{x}(t)$. Then

$$\dot{z}(t) = Az(t) + LCz(t-\tau(t)). \tag{3}$$

It is well-known that the origin of this system is GUES (Globally Uniformly Exponentially Stable) if $\overline{\tau}$ is sufficiently small.

The next section is devoted to a particular class of linear systems, namely a chain of integrators. Indeed, for these systems, the strong limitation related to the small delay constraint in the measurements vanishes.

B. Study of a particular case: chain of integrators

For some matrices A, if $\overline{\tau}$ is too large, it does not exist a matrix L such that (3) is GES. This leads us to focus our attention on an important family of systems, for which it is possible to find a matrix L so that (3) is GES (Globally Exponentially Stable) , no matter how large $\overline{\tau}$ is: in this section, we consider the particular case where $\dot{x}(t) = Ax(t)$ is a chain of integrators and $C = [1\ 0\ ...\ 0]$.

We will investigate the following problem: finding positive constants b_i such that the origin of the system:

$$\left\{ \begin{array}{rcl} \dot{z}_1(t) & = & z_2(t) - b_1 z_1(t-\tau(t)) \\ \dot{z}_2(t) & = & z_3(t) - b_2 z_1(t-\tau(t)) \\ & \vdots & \\ \dot{z}_n(t) & = & -b_n z_1(t-\tau(t)) \end{array} \right. \tag{4}$$

is GUES when $\tau(t) \in [0, \overline{\tau}]$ for all $t \geq 0$.

Let us select some positive constants k_i such that the origin of the system (47) in appendix is GES and let h^* be the corresponding constant provided by Lemma 1.1. Let $\gamma > 0$ be any constant such that

$$\frac{\overline{\tau}}{\gamma} \leq h^*. \tag{5}$$

Now, let us choose

$$b_i = \frac{k_i}{\gamma^i}\ , \quad i = 1, ..., n \tag{6}$$

and prove that, with this choice, the origin of the system (4) is GUES.

Let us apply to the system (4) a time rescaling, used for instance in [21]:

$$\xi(s) = z(\gamma s). \tag{7}$$

It gives

$$\left\{ \begin{array}{rcl} \dot{\xi}_1(s) & = & \gamma\xi_2(s) - \gamma b_1 \xi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right) \\ \dot{\xi}_2(s) & = & \gamma\xi_3(s) - \gamma b_2 \xi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right) \\ & \vdots & \\ \dot{\xi}_n(s) & = & -\gamma b_n \xi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right). \end{array} \right. \tag{8}$$

Now, the change of coordinates

$$\chi_i = \gamma^{i-1}\xi_i\ , \quad i = 1, ..., n, \tag{9}$$

used for instance in [21], gives

$$\left\{ \begin{array}{rcl} \dot{\chi}_1(s) & = & \chi_2(s) - \gamma b_1 \chi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right) \\ \dot{\chi}_2(s) & = & \chi_3(s) - \gamma^2 b_2 \chi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right) \\ & \vdots & \\ \dot{\chi}_n(s) & = & -\gamma^n b_n \chi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right). \end{array} \right. \tag{10}$$

Using (6), we obtain

$$\left\{ \begin{array}{rcl} \dot{\chi}_1(s) & = & \chi_2(s) - k_1 \chi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right) \\ \dot{\chi}_2(s) & = & \chi_3(s) - k_2 \chi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right) \\ & \vdots & \\ \dot{\chi}_n(s) & = & -k_n \chi_1\left(s - \frac{\tau(\gamma s)}{\gamma}\right). \end{array} \right. \tag{11}$$

Since the inequality (5) implies that for all $s \geq 0$, $\frac{\tau(\gamma s)}{\gamma} \leq h^*$, we deduce from Lemma 1.1 that the origin of the system (11) is GUES. It follows that the origin of the system (4) with the gains defined in (6) is GUES.

III. Extension to Nonlinear Systems

This section is devoted to three nonlinear extensions. The first one is concerned with systems possessing small globally Lipschitz nonlinear terms, the second and the last one are devoted to a particular family of feedforward systems.

A. Preliminary results

Lemma 3.1: Let k_1, ..., k_n be positive real numbers such that the origin of the system (47) in appendix is GES. Then there is a constant $\eta^* > 0$ such that the system

$$\left\{ \begin{array}{rcl} \dot{s}_1(t) & = & s_2(t) - k_1 s_1(t) + \varphi_1(s(t)) \\ \dot{s}_2(t) & = & s_3(t) - k_2 s_1(t) + \varphi_2(s(t)) \\ & \vdots & \\ \dot{s}_n(t) & = & -k_n s_1(t) + \varphi_n(s(t)) \end{array} \right. \tag{12}$$

is GES for all φ_i, $i = 1, \ldots, n$ for which there are constants $\eta_{\varphi_i} > 0$ such that

$$|\varphi_i(s)| \leq \eta_{\varphi_i} |s|\,, \forall s \in \mathbb{R}^n \tag{13}$$

and

$$\eta^* \geq \sqrt{\sum_{i=1}^{n} \eta_{\varphi_i}^2}, \tag{14}$$

where $s = (s_1, ..., s_n)^\top$.

Proof: The proof uses the quadratic Lyapunov function $\nu(s) = s^\top \mathcal{S} s$. Since $\varphi = (\varphi_1, ..., \varphi_n)^\top$ is γ_φ-Lipschitz with

$$\gamma_\varphi \leq \sqrt{\sum_{i=1}^{n} \eta_{\varphi_i}^2},$$

then using Cauchy-Schwarz inequality we can easily show that

$$\dot{\nu}(s) + \frac{\alpha}{2}\nu(s) \leq s^\top \Big[\Omega + 2\gamma_\varphi \|\mathcal{S}\|\Big] s \tag{15}$$

where

$$\Omega \triangleq (A - KC)^\top \mathcal{S} + \mathcal{S}(A - KC) + \frac{\alpha}{2}\mathcal{S}. \tag{16}$$

Since the parameters $k_1, \ldots, k_n$ are chosen such that system (47) is GES then there exist $\alpha > 0$ and $\mathcal{S} = \mathcal{S}^\top > 0$ so that $\Omega < 0$. This implies the existence of $\eta^* > 0$ small enough such that $\Omega + 2\eta^* \|\mathcal{S}\| < 0$. Hence, from (14), we deduce

$$\Omega + 2\gamma_\varphi \|\mathcal{S}\| \leq \Omega + 2\sqrt{\sum_{i=1}^{n} \eta_{\varphi_i}^2} \|\mathcal{S}\| \leq \Omega + 2\eta^* \|\mathcal{S}\| < 0$$

which leads to $\dot{\nu}(s) + \frac{\alpha}{2}\nu(s) \leq 0$ and then system (12) is GES.

■

Lemma 3.2: Let k_1,..., k_n be positive real numbers such that the origin of the system (47) is GES. Let $\eta^* > 0$ be a constant so that the system (12) is GES, with functions φ_i satisfying (13)-(14). Then there is a constant $h_{\eta^*} > 0$, a symmetric and positive definite matrix $\mathcal{S}$, and constants α_i independent from h_{η^*} such that if $h(t) \in [0, h_{\eta^*}]$ for all $t \geq 0$, then the derivative of $\nu(\varsigma) = \varsigma^\top \mathcal{S} \varsigma$ along the trajectories of

$$\left\{ \begin{array}{rcl} \dot{s}_1(t) & = & s_2(t) - k_1 s_1(t - h(t)) + \varphi_1(s(t)) + \delta_1(t) \\ \dot{s}_2(t) & = & s_3(t) - k_2 s_1(t - h(t)) + \varphi_2(s(t)) + \delta_2(t) \\ & \vdots & \\ \dot{s}_n(t) & = & -k_n s_1(t - h(t)) + \varphi_n(s(t)) + \delta_n(t), \end{array} \right. \tag{17}$$

with $\varsigma = (s_1, ..., s_n)$, satisfies

$$\begin{aligned} \dot{\nu}(t) \leq & -\alpha_1 \nu(\varsigma(t)) + \alpha_2 h_{\eta^*} \sup_{m \in [t - 2h_{\eta^*}, t]} |\varsigma(m)|^2 \\ & + \alpha_3 \sup_{m \in [t - h_{\eta^*}, t]} |\Delta(m)|^2, \end{aligned} \tag{18}$$

with $\Delta = (\delta_1, ..., \delta_n)$.

Proof: The proof is based on Lemma 3.1 and the well known Razumikhin's theorem to get (18). For more details, see Lemma 1.1 in Appendix. Indeed, Lemma 3.2 is a generalization of Lemma 1.1 to nonlinear systems. ■

B. Observer design for particular nonlinear systems

By analogy with the linear case, we show now that under some assumption on the structure of the nonlinearities, we can allow an arbitrarily long delay in the output measurement.

Let us consider the system described by:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + G(t)u(t) + \varphi(x(t)) \\ y(t) & = & Cx(t - \tau(t)), \end{array} \tag{19}$$

where A is such that $\dot{x} = Ax$ is a chain of integrator, $C = [1\ 0\ ...\ 0]$ and where $G(.)$ is a piecewise continuous function. Notice that the pair (A, C) is detectable.

We assume that $\tau(.)$ is piecewise continuous and that there is $\bar{\tau} > 0$ such that $\tau(t) \in [0, \bar{\tau}]$ for all $t \geq 0$.

Let us introduce the following candidate observer:

$$\begin{aligned} \dot{\hat{x}}(t) = & A\hat{x}(t) + G(t)u(t) + \varphi(\hat{x}(t)) \\ & + L[y(t) - C\hat{x}(t - \tau(t))]. \end{aligned} \tag{20}$$

Let $z = x - \hat{x}$. Then

$$\dot{z}(t) = Az(t) - LCz(t - \tau(t)) + \delta\varphi(x, \hat{x}), \tag{21}$$

where $\delta\varphi(x, \hat{x}) = \varphi(x) - \varphi(\hat{x})$ satisfies:

$$|\delta\varphi_i(x, \hat{x})| \leq \eta_{\varphi_i} |z|, \quad \forall x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^n. \tag{22}$$

As in the linear case, one can prove that, under the conditions of Lemma 3.1, the origin of the system (21) is GUES if $\bar{\tau}$ is sufficiently small. In order to tolerate an arbitrarily large delay in the measurements, we need additional assumptions on the nonlinear functions φ_i.

1) First case (a first class of feedforward systems): Assume that the nonlinear functions satisfy the following conditions:

$$\frac{\partial \varphi_i}{\partial s_j}(.) \equiv 0, \forall j = 1, \ldots, i + 1. \tag{23}$$

Let us select some positive constants k_i and η^* ensuring that the origin of the system (12) is GES, and let h_{η^*} be the corresponding constant provided by Lemma 3.2. Let $\gamma > 0$ be any constant such that

$$\frac{\overline{\tau}}{\gamma} \leq h_{\eta^*}. \tag{24}$$

Now, let us choose

$$L = \begin{pmatrix} \frac{k_1}{\gamma^1} & \ldots & \frac{k_n}{\gamma^n} \end{pmatrix}^\top. \tag{25}$$

As in the linear case, we introduce the time rescaling

$$\xi(s) = z(\gamma s).$$

Hence the change of coordinates

$$\chi_i = \gamma^{i-1} \xi_i\ , \quad i = 1, ..., n, \tag{26}$$

leads to

$$\begin{array}{rcl} \dot{\chi}(s) & = & A\chi(s) - KC\chi\left(s - \frac{\tau(s)}{\gamma}\right) \\ & & + \Gamma(\gamma)\delta\varphi\Big(x(\gamma s), \hat{x}(\gamma s)\Big), \end{array} \tag{27}$$

where $K = (k_1 \ldots k_n)^\top$ and $\Gamma(\gamma) = \mathrm{diag}(\gamma, \ldots, \gamma^n)$.

It is easy to see that the property (23) implies that

$$\left|\Gamma(\gamma)\delta\varphi\Big(x(\gamma s), \hat{x}(\gamma s)\Big)\right| \leq \frac{1}{\gamma}\sqrt{\sum_{i=1}^{n-2}\eta_{\varphi_i}^2}|\chi(s)|. \tag{28}$$

Now, we select

$$\gamma \geq \max\left\{\frac{\overline{\tau}}{h_{\eta^*}}, \frac{1}{\eta^*}\sqrt{\sum_{i=1}^{n-2}\eta_{\varphi_i}^2}\right\} \tag{29}$$

and get $\frac{\tau(\gamma s)}{\gamma} \leq h_{\eta^*}$ and $\eta^* \geq \frac{1}{\gamma}\sqrt{\sum_{i=1}^{n-2}\eta_{\varphi_i}^2}$. Consequently, we deduce from Lemma 3.2 that the origin of the system (27) is GUES. It follows that the origin of the system (21) is GUES.

2) Second case (a second class of feedforward systems): The condition (23) can be slightly relaxed, but then the determination of some constants is slightly more difficult, as we shall see in this section. In fact, we can tolerate the component x_{i+1} inside the nonlinear function φ_i, in other words, we can replace the condition (23) by

$$\frac{\partial\varphi_i}{\partial s_j}(.) \equiv 0, \forall j = 1, \ldots, i. \tag{30}$$

Now, one can follow exactly the same developments as in the previous section, except that, in this case, the parameters k_i are not to be computed so that the matrix $A-KC$ is Hurwitz, but have to be computed by solving a set of LMIs (Linear Matrix Inequalities) independent of γ, by exploiting for instance the LPV/LMI approach given in [22].

Indeed, the nonlinear term $\delta\varphi(x(\gamma s), \hat{x}(\gamma s))$ can be written as:

$$\begin{aligned}
&\Gamma(\gamma)\delta\varphi\Big(x(\gamma s), \hat{x}(\gamma s)\Big) \\
&= \underbrace{\sum_{i=1}^{n-2}\sum_{j=i+2}^{n}\frac{\gamma i}{\gamma^{j-1}}\frac{\partial\varphi_i}{\partial x_j}(\bar{\nu}_i(s))\mathrm{e}_n(i)\mathrm{e}_n^\top(j)\chi(s)}_{\Delta\bar{\varphi}\left(x(\gamma s),\hat{x}(\gamma s)\right)} \\
&\quad + \sum_{i=1}^{n-1}\overbrace{\frac{\partial\varphi_i}{\partial x_{i+1}}(\bar{\nu}_{i+1}(s))}^{\varrho_i(s)}\mathrm{e}_n(i)\mathrm{e}_n^\top(i)\chi(s),
\end{aligned} \tag{31}$$

where $\mathrm{e}_n(i) = \big(\underbrace{0, \ldots, 0, \overbrace{1}^{i\text{ th}}, 0, \ldots, 0}_{n\text{ components}}\big)^T \in \mathbb{R}^n, n \geq 1$ is a vector of the canonical basis of $\mathbb{R}^n$.

Using the notations

$$\mathcal{A}\Big(\varrho(s)\Big) = A + \sum_{i=1}^{n-1}\varrho_i(s)\mathrm{e}_n(i)\mathrm{e}_n^\top(i), \tag{32a}$$

$$\varrho(s) = \big(\varrho_i(s), \ldots, \varrho_{n-1}(s)\big) \tag{32b}$$

then the system (27) becomes

$$\dot{\chi}(s) = \mathcal{A}\Big(\varrho(s)\Big)\chi(s) - K\chi\left(s - \frac{\tau(s)}{\gamma}\right) + \Delta\bar{\varphi}\Big(x(\gamma s), \hat{x}(\gamma s)\Big). \tag{33}$$

From (13), it follows that the parameter $\varrho(.)$ is bounded from below and above as $-\eta_i \leq \varrho_i(s) \leq \eta_i$. This means that the affine matrix function $\mathcal{A}\Big(\varrho(s)\Big)$ belongs to an hyper rectangle $\mathfrak{R}$ for which the set of vertices is defined by

$$\mathcal{P}_{\mathfrak{R}} = \Big\{\mathcal{A}\Big(\rho\Big) \in \mathcal{M}_n(\mathbb{R}) : \rho_i \in \{-\eta_i, \eta_i\}, i = 1, \ldots, n-1\Big\}. \tag{34}$$

We can easily check that

$$\left|\Delta\bar{\varphi}\Big(x(\gamma s), \hat{x}(\gamma s)\Big)\right| \leq \frac{1}{\gamma}\left(\sum_{i=1}^{n-2}\eta_{\varphi_i}(n-i-1)\right)|\chi(s)|, \quad \forall\chi(s). \tag{35}$$

Taking into account (28), we can write:

$$\left|\Delta\bar{\varphi}\Big(x(\gamma s), \hat{x}(\gamma s)\Big)\right| \leq \frac{1}{\gamma}\Theta_{n,\eta}\left|\chi(s)\right| , \forall\chi(s). \tag{36}$$

where

$$\Theta_{n,\eta} = \min\left\{\sqrt{\sum_{i=1}^{n-2}\eta_{\varphi_i}^2}, \sum_{i=1}^{n-2}\eta_{\varphi_i}(n-i-1)\right\}.$$

Introduce the following lemma.

Lemma 3.3: Let $k_i, i = 1, \ldots, n$ be positive constants such that

$$\Big(\mathcal{A}-KC\Big)^\top S + S\Big(\mathcal{A}-KC\Big) + \frac{\alpha}{2}S < 0, \ \forall\mathcal{A} \in \mathcal{P}_{\mathfrak{R}}, \tag{37}$$

where α is a positive scalar and S is a symmetric positive definite matrix. Let $\eta^* > 0$ be such that the system

$$\dot{\zeta}(t) = \Big(\mathcal{A}\big(\varrho(t)\big) - KC\Big)\zeta(t) + \Psi\big(\zeta(t)\big), \tag{38}$$

with

$$\left|\Psi\big(\zeta(t)\big)\right| \leq \eta^*\left|\zeta(t)\right| , \forall\zeta(t) \tag{39}$$

is GES. Then, there exists a constant $h_{\eta^*} > 0$ such that the origin of the system

$$\dot{\zeta}(t) = \mathcal{A}\big(\varrho(t)\big)\zeta(t) - KC\zeta\big(t - h(t)\big) + \Psi\big(\zeta(t)\big) \tag{40}$$

is GUES provided $h(t) \leq h_{\eta^*}$ for all $t \geq 0$.

Proof: This Lemma is an extension of Lemma 3.2 to systems with non constant state matrix A, namely $\mathcal{A}\big(\varrho(t)\big)$. It is so obvious from the convexity principle that if (37) then system (38) is GES for η^* small enough. Hence, as in Lemma 3.2, Razumikhin's theorem concludes the proof and provides the existence of a constant $h_{\eta^*} > 0$ such that the origin of the system (40) is GUES for all h such that $h(t) \leq h_{\eta^*}$. ■

Now, we can state the following proposition.

Proposition 3.4: Let $k_i, i = 1, \ldots, n$ and η^* be positive real numbers such that (37) and (39) are satisfied. Let h_{η^*} be a positive scalar such that the origin of (40) is GUES when $h(.)$ is such that $h(t) \leq h_{\eta^*}$ for all $t \geq 0$. Then, the origin of the system (33) is GUES for all τ such that $\tau(t) \leq \bar{\tau}$ for all $t \geq 0$ if

$$\gamma \geq \max\left\{\frac{\overline{\tau}}{h_{\eta^*}}, \frac{1}{\eta^*}\min\left(\sqrt{\sum_{i=1}^{n-2}\eta_{\varphi_i}^2}, \sum_{i=1}^{n-2}\eta_{\varphi_i}\left(n-i-1\right)\right)\right\}. \tag{41}$$

Proof: The proof is straightforward. Indeed, according to Lemma 3.3 and since $\bar{\varphi}$ satisfies (36), we deduce that system (33) is GUES if

1) $\frac{\tau(s)}{\gamma} = \frac{\overline{\tau}}{\gamma} \leq h_{\eta^*}$;
2) $\eta^* \geq \frac{1}{\gamma}\Theta_{n,\eta}$

which mean that system (33) is GUES if (41) holds. ■

IV. Illustrative Example

This section is devoted to a numerical example to illustrate the validity and effectiveness of the proposed small gain approach. Consider a fifth dimensional academic example under the form (19) with $G = 0$ and:

$$\varphi(x) = \begin{pmatrix} \sin(x_3) \\ \cos(x_4) \\ \sin(x_5) \\ 0 \\ 0 \end{pmatrix}. \tag{42}$$

It is quite clear that $\varphi(.)$ satisfies the condition (23). It is also easy to see that $\eta_{\varphi_i} = 1$.

To find a η^* provided by Lemma 3.1, we can use the following standard LMI technique [22]:

$$\begin{bmatrix} (A-KC)^\top S + S(A-KC) + \frac{\sigma}{2}S + \eta^* I_n & S \\ S & -\eta^* I_n \end{bmatrix} < 0. \tag{43}$$

After solving first the LMI

$$(A-KC)^\top S + S(A-KC) + \frac{\sigma}{2}S < 0 \tag{44}$$

for $\sigma = 1$, we found the solution

$$K = \begin{pmatrix} 4.0444 \\ 13.8753 \\ 17.9352 \\ 13.8180 \\ 4.3906 \end{pmatrix}, \tag{45}$$

which ensures that the origin of the system (47) is GES. Using this solution K, the optimal value of η^* given by the LMI (43), with the decision variables S and η^*, for which the origin of the system (12) is GES is $\eta^* = 0.7468$.

Now, from Lemma 3.2, there exists $h_{\eta^*} > 0$ so that the origin of the system (17) is GUES. The corresponding maximum value provided by Matlab simulation is $h_{\eta^*} = 0.0131$. Hence, using the time rescaling $t \mapsto \gamma s$ and the small gain approach proposed in this paper, we deduce that if

$$L_i = \frac{k_i}{\gamma^i}$$

where

$$\gamma \geq \max\left\{\frac{\overline{\tau}}{h_{\eta^*}}, \frac{1}{\eta^*}\sqrt{\sum_{i=1}^{3}\eta_{\varphi_i}^2}\right\} \tag{46}$$

then the origin of the system (21) is GUES for all τ such that $\tau(t) \leq \bar{\tau} \leq \gamma h_{\eta^*}$ for all $t \geq 0$. Figure 1 shows some simulation scenarios for different values of γ. As shown in Figure 1, the small gain approach allows long delay in the output measurements at the cost of weak convergence rate. It is quite clear from Figure 1(b) that the estimation errors do not converge to zero if the small gain parameter γ does not satisfy (46). On the other hand, if γ satisfies (46), then the estimation errors converge exponentially to the origin, but the convergence rate is low, as can be seen in Figures 1(c)- 1(d).

V. Conclusion

This paper is concerned with state observer design problems for systems with delayed output measurements. We provided a new small gain approach, which guarantees exponential convergence of the observers for arbitrarily long delay in the measurements for a particular family of feedforward systems. A numerical example was presented to show the efficiency of the proposed small gain approach in the presence of large delay in the outputs. Much remains to be done, in particular, we will study the problem of establishing semi-global asymptotic output feedback results in the case where the systems are in feedforward form but not globally Lipschitz.

Appendix

Lemma 1.1: Let k_1, ... , k_n be positive real numbers such that the origin of the system

$$\left\{\begin{array}{rcl} \dot{s}_1(t) & = & s_2(t) - k_1 s_1(t) \\ \dot{s}_2(t) & = & s_3(t) - k_2 s_1(t) \\ & \vdots & \\ \dot{s}_n(t) & = & -k_n s_1(t) \end{array}\right. \tag{47}$$

is GES. Then there is a constant $h^* > 0$, a symmetric and positive definite matrix $\mathcal{S}$ and constants α_i independent from h^* such that if $h(t) \in [0, h^*]$ for all $t \geq 0$, then the derivative of $\nu(\varsigma) = \varsigma^\top \mathcal{S} \varsigma$ along the trajectories of

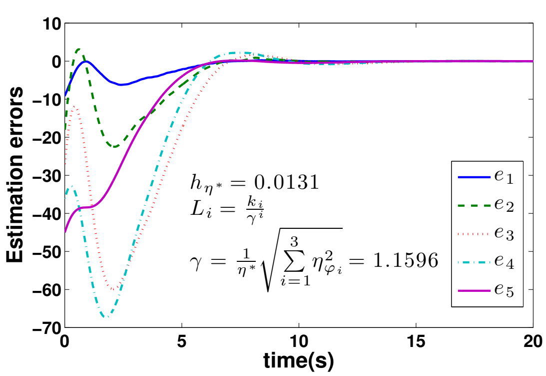
$$\left\{\begin{array}{rcl} \dot{s}_1(t) & = & s_2(t) - k_1 s_1(t-h(t)) + \delta_1(t) \\ \dot{s}_2(t) & = & s_3(t) - k_2 s_1(t-h(t)) + \delta_2(t) \\ & \vdots & \\ \dot{s}_n(t) & = & -k_n s_1(t-h(t)) + \delta_n(t), \end{array}\right. \tag{48}$$

with $\varsigma = (s_1, ..., s_n)$, satisfies

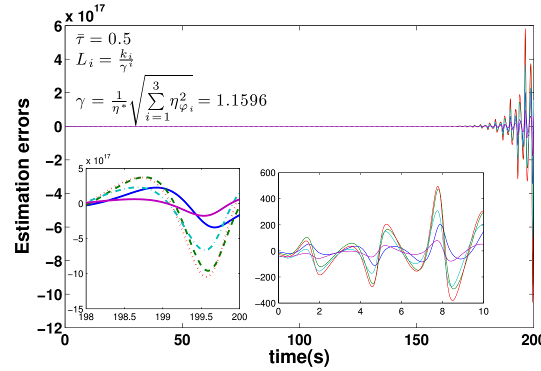
$$\dot{\nu}(t) \leq -\alpha_1 \nu(\varsigma(t)) + \alpha_2 h^* \sup_{m\in[t-2h^*,t]} |\varsigma(m)|^2 + \alpha_3 \sup_{m\in[t-h^*,t]} |\Delta(m)|^2 \tag{49}$$

with $\Delta(t) = (\delta_1, ..., \delta_n)$.

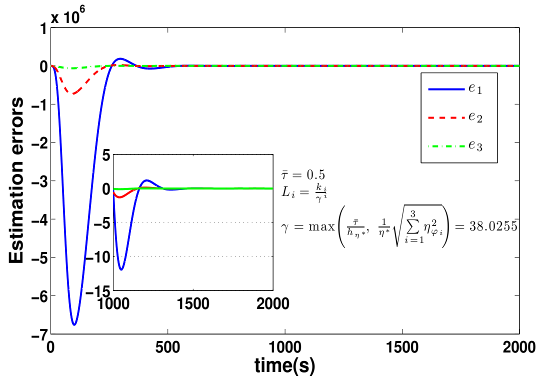
Proof: The proof is omitted. ■



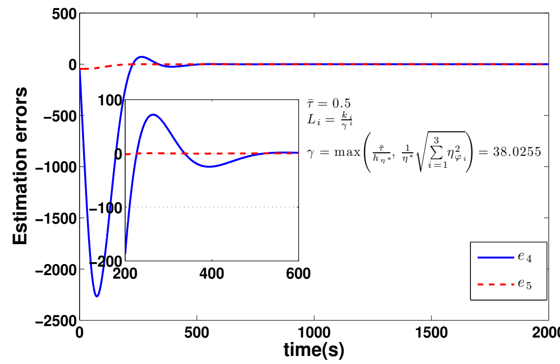
(a) Estimation errors for a small delay.



(b) Estimation errors for a long delay without (46).



(c) Behavior of e_1, e_2, e_3.



(d) Behavior of e_4 and e_5.

Fig. 1. Behavior of the estimation errors using small gain approach.

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